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AN ACCURATE ELEMENTARY STATIC THEORY OF LAMINATED THERMOELASTIC BEAMS

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Abstract—We show how, starting from an elementary, one-dimensional thermoelastic theory of beams, we may use an asymptotic analysis to approximate the temperature distribution in thin, orthotropic laminated beams and obtain an explicit, mean square error estimate using the hypercircle method. The temperature distribution gives rise to stresses that also are approximated using the asymptotic–hypercircle approach, as we illustrate in the analysis of a simple laminated beam. The relative error in both the temperature and stress distributions can be reduced to the order of magnitude of an arbitrary power of the ratio of the beam thickness to a characteristic length associated with the external thermal load. The method does not require any *a priori* assumptions regarding the thickness distributions of the temperature, displacements or stresses.

INTRODUCTION

Building on the work of Rychter (1988), Duva and Simmonds (to appear) have shown that an elementary beam theory solution is all that is needed to generate two-dimensional strains and stress fields of arbitrary mean square accuracy in an orthotropic beam, provided that we demand no more detail at the ends of the beam than the shear stress resultant or the average vertical displacement and the bending moment or the average rotation. The key to obtaining a relatively small error in the two-dimensional strain field inferred from one-dimensional beam theory is to construct a statically admissible strain field and a kinematically admissible strains field whose through-thickness distributions are nearly equal. The program for this construction laid out by Duva and Simmonds was shown to be applicable even if the shear modulus is very much smaller than the axial extensional modulus.

In this paper we modify the program for obtaining accurate two-dimensional strain and stress fields in an orthotropic beam subject to mechanical loads to obtain accurate two-dimensional temperature, stress and strain fields in an orthotropic beam subject to a static thermal load. To show that our procedure applies to stratified media as well, we end the paper by analyzing a simple laminated beam in which significant thermal stresses arise.

As temperature can be determined independently of the deformation, we can construct, by inference from the axial distribution of temperature and the transverse temperature gradient delivered by *elementary beam theory*, approximate two-dimensional temperature fields that are polynomials of degree $2N+1$ in the thickness coordinate. Their relative mean square error is of order $(H/l)^{2N+1}$, where H is the thickness of the beam and l is a computable characteristic length associated with the thermal load. For beams with large axial conductivities, the relative mean square error is of order $(HL/l^2)^N$, where L is the length of the beam. In obtaining these error estimates we assume that at the ends of the beam the prescribed temperatures or heat fluxes are compatible with the fields we construct. If not, then a full two-dimensional treatment of the ends must be considered, as discussed, for example, in the Appendix to Chapter 10 of Boley and Weiner (1960), and the solution we construct is valid only in the interior of the beam away from the ends. The boundary layer fields at the beam ends may be of primary importance in predicting delamination failure of a layered beam, as noted by Chen *et al.* (1982).

In general, the inferred two-dimensional temperature field will produce stresses due to kinematic conditions at a boundary or an interface in a laminated beam, and due to the inhomogeneity of the temperature field itself. Once the temperature field is known, the program laid out by Duva and Simmonds can be used to generate stress and strain fields of any desired accuracy (although not in excess of the accuracy to which the temperature field is determined). This procedure is similar to that put forward by Boley and Weiner, differing in that Boley and Weiner take the temperature field as given and do not obtain explicit error estimates.

THE GOVERNING EQUATIONS

Let $Oxyz$ denote a fixed, right-handed Cartesian reference frame and consider a rectangular beam that, when undeformed, occupies the region $0 \leqslant x \leqslant L, |y| \leqslant D, |z| \leqslant H$; see Fig. 1. We assume that the beam is built-in at $x = 0$ and traction-free at $x = L$. The upper and lower surfaces of the beam are traction-free, there are no body forces, the broad faces of the beam are insulated, and the upper and lower surfaces of the beam are held at incremental temperatures

$$T(x, \pm H) = \pm T_0\theta(x/L), \tag{1}$$

where T measures temperature above some constant absolute reference temperature T_R and θ is a prescribed dimensionless incremental temperature. As stated above, we assume the thermal boundary conditions at the ends of the beam are met, so they need not be made explicit. We further assume that the beam is homogeneous and orthotropic with material axes aligned with the reference frame, and that linear elastic plane stress theory and the linear theory of heat conduction in the xz-plane apply. This configuration was chosen for definiteness and simplicity, although any configuration could be used.

The field equations consist of the mechanical and thermal equilibrium equations, the compatibility equation for strains, the stress–strain relations, and Fourier's Law relating the heat flux and the temperature. We consider only the thermal problem in detail, as the mechanical problem has been discussed by Duva and Simmonds. First, we choose a *kinematically admissible* temperature field T^K that satisfies the prescribed temperatures on the upper and lower faces of the beam ($z = \pm H$). Thermal equilibrium is satisfied by introducing a *flux potential* P so that the Cartesian components of the *statically admissible* heat flux $\mathbf{q}^S$ are given by

$$q_x^S = P_{,z} \quad q_z^S = -P_{,x}, \tag{2}$$

where a subscript preceded by a comma indicates partial differentiation. To incorporate Fourier's Law we define

$$\Delta q_x \equiv P_{,z} + k_x T_{,x}^K = q_x^S - q_x^K \tag{3a}$$

$$\Delta q_z \equiv -P_{,x} + k_z T_{,z}^K = q_z^S - q_z^K, \tag{3b}$$

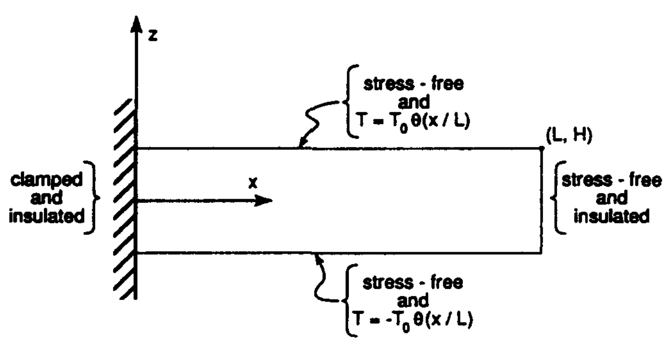

Fig. 1. An undeformed rectangular orthotropic beam of length L and height $2H$ subjected to the boundary conditions as shown. It is assumed that a state of plane stress holds and that there is no heat flow perpendicular to the xz-plane.

where k_x and k_z are the conductivities in the axial and thickness directions, respectively, and $\mathbf{q}^K = (q_x^K, q_z^K)$ is a *kinematically admissible* heat flux.

According to the hypercircle method of Prager and Synge (1947), $\mathbf{q}^S$ and $\mathbf{q}^K$ must lie on a hyperplane and hypersphere, respectively, in a function space of infinite dimension. As the actual heat flux $\mathbf{q}$ is both statically and kinematically admissible, it must lie on the hypercircle defined by the intersection of the hyperplane and hypersphere mentioned above. Further, the center of the hypercircle is given by $\frac{1}{2}(\mathbf{q}^S+\mathbf{q}^K)$. Thus, the radius of the hypercircle,

$$\|\mathbf{q}-\tfrac{1}{2}(\mathbf{q}^K+\mathbf{q}^S)\| = \|\Delta\mathbf{q}\|, \tag{4}$$

is the magnitude of the error when the hypercircle center is taken as an approximation to $\mathbf{q}$, where

$$\|\mathbf{q}\|^2 = \int_{-H}^{H}\int_0^L (q_x^2+q_z^2)\,\mathrm{d}x\,\mathrm{d}z \tag{5}$$

is the square of the Dirichlet norm. The actual temperature field is both kinematically and statically admissible and satisfies the steady-state energy equation

$$k_x T_{,xx}+k_z T_{,zz} = 0. \tag{6}$$

We will show how, by starting with the lowest order approximation to T (analogous to the elementary beam theory solution in the mechanical context) and performing simple integrations through the thickness, we may construct an approximation to T with a formal error as small as we please.† We then use (3a, b) to construct approximate heat flux potentials such that the *computable relative error*, $\|\Delta\mathbf{q}\|/\|\mathbf{q}^{(0)}\|$, is bounded by a constant time as high a power of H/l as we please. The term $\mathbf{q}^{(0)}$ is the lowest order approximation to the actual heat flux $\mathbf{q}$, which is unknown and thus unavailable for computing the relative error.

TEMPERATURES, STRESSES AND ERROR ESTIMATES IN A HOMOGENEOUS BEAM

To emphasize the key points, we confine our analysis in this section to a homogeneous (i.e. non-laminated) orthotropic beam. In the next section we generalize to a layered beam. We introduce the dimensionless variables and parameters

$$\xi = x/L, \quad \zeta = z/H, \quad T = T_0\tau, \quad \varepsilon = H/L,$$

and rewrite (6) and (1) as

$$\tau_{,\zeta\zeta} = -\varepsilon^2\bar{k}\tau_{,\xi\xi}, \quad \bar{k} = k_x/k_z, \tag{7a}$$

$$\tau(\xi, \pm 1) = \pm\theta(\xi). \tag{7b}$$

In an isotropic material $\bar{k} = 1$. We assume that the prescribed temperatures or heat fluxes at the ends of the beam are odd is z; thus, the dimensionless temperature τ will be odd in ζ. Integrating (7a) twice with respect to ζ gives

† Alternatively, we could start with the "compatibility" condition $k_x P_{,xx}+k_z P_{,zz} = 0$ satisfied by the actual flux potential, just as in the mechanical context we started with the strain compatibility equation satisfied by the Airy stress function. This equation would be preferable to (6) if heat fluxes rather than temperatures were prescribed on the faces of the beam.

$$\tau(\xi,\zeta) = A(\xi)\zeta - \varepsilon^2\tilde{k}\left[\int_0^\zeta (\zeta - s)\tau(\xi,s)\,\mathrm{d}s\right]_{,\xi\xi}, \tag{8}$$

where $A(\xi)$ is a function of integration. Imposition of the boundary conditions (7b) gives

$$\tau(\xi,\zeta) = \tau^{(0)} + \varepsilon^2\tilde{k}I\tau_{,\xi\xi}, \tag{9}$$

where

$$\tau^{(0)} = \theta(\xi)\zeta$$

and

$$I\tau = \int_0^\zeta (1-\zeta)s\tau(\xi,s)\,\mathrm{d}s + \int_\zeta^1 (1-s)\zeta\tau(\xi,s)\,\mathrm{d}s.$$

Note that in the elementary, one-dimensional theory of thermoelastic beam(shell)s developed by Libai and Simmonds (1988), the average axial stress σ due to heating in a homogeneous, isotropic beam is

$$\sigma = 2EH(e - \alpha\hat{\tau})$$

where e is the average axial strain, α is the coefficient of thermal expansion, E is the extensional modulus, and $\hat{\tau}$ is the *mean temperature increment* ($T = \hat{\tau} + T_R$). From their equation (T.9) and our (7b), $\hat{\tau} = 0$. The bending moment due to heating is

$$M = (2/3)EH^3(\kappa + \alpha\Delta),$$

where κ is the curvature and Δ is the *transverse temperature gradient*

$$\Delta = \frac{T_0\theta}{H}\left[1 - \underline{\left(\frac{T_0\theta}{T_R}\right)^2}\right],$$

which follows from their equation (T.9). As the underlined term disappears in a linear theory, $\tau^{(0)}$ can be expressed in terms of Δ as $\Delta H\zeta/T_0$. As will be shown, the stresses due to heating are zero; thus, $e = 0$ and $\kappa = -\alpha\Delta$.

If we seek an approximate solution of the form

$$\tau = \tau^{(0)} + \varepsilon^2\tau^{(1)} + \cdots + \varepsilon^{2N}\tau^{(N)}, \tag{10}$$

then (9) yields the recurrence relation

$$\tau^{(n)} = \tilde{k}I\tau^{(n-1)}_{,\xi\xi}, \quad n = 1, 2, \ldots, N. \tag{11}$$

Application of this relation gives, for $n = 1$,

$$\tau^{(1)} = \tilde{k}\theta''(\xi)(\zeta - \zeta^3)/6, \tag{12a}$$

and, for arbitrary n,

$$\tau^{(n)} = \theta^{[2n]}(\xi)\mathscr{P}_{[2n+1]}(\zeta). \tag{12b}$$

Here, the superscript $[2n]$ on θ indicates $2n$ differentiations and $\mathscr{P}_{[2n+1]}$ is an odd polynomial in ζ of order $2n+1$. We note that the approximation

$$\tau = \tau^{(0)} + \varepsilon^2 \tau^{(1)} = \theta(\xi)\zeta + \bar{k}\theta''(\xi)(\zeta - \zeta^3)/6$$

gives rise to zero local heat fluxes at the beam ends only if θ' and θ''' are zero there. However, the overall heat flux at the ends of the beam is zero in any case.

The dimensionless temperature field τ gives rise to a kinematically admissible heat flux $\mathbf{q}^K$. This temperature field has the added virtue—because it satisfies (6) asymptotically to order ε^{2N+2}—that, in conjunction with (3a, b), it can be used to construct a statically admissible heat flux $\mathbf{q}^S$ that is near $\mathbf{q}^K$. We introduce a dimensionless heat flux potential

$$\rho = P\varepsilon / k_z T_0$$

so that (3a, b) take the form

$$\Delta q_x = q_0(\varepsilon^{-2}\rho_{,\zeta} + \bar{k}\tau_{,\xi}), \tag{13a}$$

$$\Delta q_z = q_0 \varepsilon^{-1}(-\rho_{,\xi} + \tau_{,\zeta}), \tag{13b}$$

where $q_0 = k_z T_0 / L$.

We now make the approximation

$$\rho = \rho^{(0)} + \varepsilon^2 \rho^{(1)} + \cdots + \varepsilon^{2N} \rho^{N}. \tag{14}$$

Noting (13a, b), we make $\|\Delta \mathbf{q}\|$ small by setting

$$\rho^{(n)}_{,\zeta} = -\bar{k}\tau^{(n-1)}_{,\xi}, \tag{15a}$$

$$\rho^{(n)}_{,\xi} = \tau^{(n)}_{,\zeta}, \tag{15b}$$

where $n = 0, 1, \ldots, N$ and $\tau^{(-1)}$ is taken to be zero. Integrating (15a) with respect to ζ gives

$$\rho^{(n)} = A_n(\xi) - \bar{k}\left[\int_0^{\zeta} \tau^{(n-1)}(\xi, s)\,\mathrm{d}s\right]_{,\xi}, \tag{16}$$

where $A_n(\xi)$ is a function of integration. Differentiating (16) with respect to ξ and substituting into (15b), we obtain

$$A_n'(\xi) = \tau^{(n)}_{,\zeta} + \bar{k}\left[\int_0^{\zeta} \tau^{(n-1)}(\xi, s)\,\mathrm{d}s\right]_{,\xi\xi}. \tag{17}$$

Using (11) to rewrite $\tau^{(n)}$ in terms of $\tau^{(n-1)}$ (for $n \geqslant 1$) and integrating (17) once with respect to ξ, we get

$$A_n(\xi) = \bar{k}\left[\int_0^{1} (1-s)\tau^{(n-1)}(\xi, s)\,\mathrm{d}s\right]_{,\xi}, \tag{18}$$

where the integration constant which has no physical meaning has been ignored. For $n = 0$, (16) gives

$$\rho^{(0)} = \int_0^{\xi} \theta(s)\,\mathrm{d}s \tag{19}$$

where, again, the constant of integration is ignored.

To determine the size of $\|\Delta \mathbf{q}\|$, given an N-term expansion for τ, we need not explicitly calculate the $\rho^{(n)}$ if they can be computed in principle as outlined above. Neither need we calculate them to make the solution for all quantities of interest explicit. So, instead of filling out the expansion for ρ, we move immediately to the computation of the size of the relative error.

Suppose $\rho^{(0)}$ through $\rho^{(N)}$ are calculated from (16) and (18) so that (15a, b) are satisfied. Then (13a, b) give

$$\Delta q_x = q_0 \bar{k} \varepsilon^{2N} \tau^{(N)}_{,\xi} \tag{20a}$$

$$\Delta q_z = 0, \tag{20b}$$

where $\tau^{(N)}$ has the form given in (12b). Substituting into (5) we find

$$\|\Delta \mathbf{q}\|^2 = \varepsilon^{4N} H L q_0^2 \bar{k}^2 \int_0^1 (\theta^{[2N+1]})^2 \, d\xi \int_{-1}^1 (\mathscr{P}_{[2N+1]})^2 \, d\zeta. \tag{21}$$

To simplify (21), which expresses the mean square error in terms of high order derivatives of the thermal load θ, we define the characteristic length l associated with θ by

$$\left(\frac{l}{L}\right)^{2M} \equiv \frac{\int_0^1 \theta^2 \, d\xi}{\int_0^1 (\theta^{[M]})^2 \, d\xi} \tag{22}$$

so that rapid variations in θ render l/L small. Then,

$$\|\Delta \mathbf{q}\|^2 = \varepsilon^{4N} H L q_0^2 \bar{k}^2 \int_{-1}^1 \mathscr{P}^2_{[2N+1]} \, d\zeta \int_0^1 \theta^2 \, d\xi \, (L/l)^{4N+2}. \tag{23}$$

To compute the relative error we note that

$$\mathbf{q}^{(0)} = q_0(\bar{k}\tau^{(0)}_{,\xi}, \varepsilon^{-1}\tau^{(0)}_{,\zeta}) = q_0(\bar{k}\theta'\zeta, \varepsilon^{-1}\theta),$$

and hence

$$\|\mathbf{q}^{(0)}\|^2 = 2\varepsilon^{-2} H L q_0^2 \int_0^1 \theta^2 \, d\xi + O(1), \tag{24}$$

so the relative error is

$$\frac{\|\Delta \mathbf{q}\|}{\|\mathbf{q}^{(0)}\|} = O\left(\frac{H}{l}\right)^{2N+1}. \tag{25}$$

For example, if $\theta = \cos(\pi\xi)$, then, to first order,

$$\tau = \zeta \cos(\pi\xi) - \varepsilon^2 \bar{k} \pi^2 \cos(\pi\xi)(\zeta - \zeta^3/6).$$

The error in the heat flux can be computed from (23) as

$$\|\Delta \mathbf{q}\|^2 = \tfrac{1}{2}\varepsilon^4 (L/l)^6 H L q_0^2 \bar{k}^2 \int_{-1}^1 \mathscr{P}_3^2 \, d\zeta$$

where $\mathscr{P}_3 = \bar{k}^2(7\zeta/30 - \zeta^3/3 - \zeta^5/10)/12$. As $\|\mathbf{q}^{(0)}\| = \varepsilon^{-2} H L q_0$, we have

$$\frac{\|\Delta \mathbf{q}\|}{\|\mathbf{q}^{(0)}\|} \approx 0.0065\tilde{k}^2\left(\frac{H}{l}\right)^3.$$

Note that from (22) with $M = 3$ we obtain $L = l$, as expected for the given thermal loading.

Finally, we note that if $\tilde{k} = \varepsilon^{-1}\hat{k}, \hat{k} = O(1)$, as would be the case with a composite reinforced in the axial direction with continuous and highly conductive fibers, then (10) and (14) must be replaced with

$$\tau = \tau^{(0)} + \varepsilon\tau^{(1)} + \cdots + \varepsilon^N\tau^{(N)} \tag{26a}$$

$$\rho = \rho^{(0)} + \varepsilon\rho^{(1)} + \cdots + \varepsilon^N\rho^{(N)}. \tag{26b}$$

The analysis required to determine the size of the relative error if $\tilde{k} = O(\varepsilon^{-1})$ is as given above, except that now

$$\mathbf{q}^{(0)} = q_0\varepsilon^{-1}(\hat{k}\theta'\zeta, \theta),$$

so that

$$\|\mathbf{q}^{(0)}\|^2 = \varepsilon^{-2}HLq_0^2[(2/3)\hat{k}^2(L/l)^2 + 2]\int_0^1 \theta^2\,\mathrm{d}\xi.$$

Hence,

$$\frac{\|\Delta \mathbf{q}\|}{\|\mathbf{q}^{(0)}\|} = O\left(\frac{HL}{l^2}\right)^N. \tag{27}$$

A statically admissible stress field resulting from heating is most easily obtained through the introduction of the Airy stress function F that must satisfy the strain compatibility relation

$$\frac{1}{E_x}F_{,zzzz} + \left[\frac{1}{G} - \frac{2\nu}{E_z}\right]F_{,xxzz} + \frac{1}{E_z}F_{,xxxx} = -\alpha_x T_{,zz} - \alpha_z T_{,xx}, \tag{28a}$$

which follows from equations (A1)–(A3) of the Appendix. Here E_x, E_z and G are elastic moduli, ν is Poisson's ratio, and α_x and α_z are coefficients of thermal expansion. For isotropic materials, (28a) reduces to

$$\nabla^4 F = -\alpha E \nabla^2 T. \tag{28b}$$

Introducing a dimensionless stress function through

$$f = F/L^2 E_x \alpha_z T_0, \tag{29}$$

we recast (28a) in the form

$$f_{,\zeta\zeta\zeta\zeta} = -2\varepsilon^2\bar{E}f_{,\zeta\zeta\xi\xi} - \varepsilon^4\tilde{E}f_{,\xi\xi\xi\xi} - \varepsilon^2\hat{\alpha}\tau_{,\zeta\zeta} - \varepsilon^4\tau_{,\xi\xi}, \tag{30}$$

where

$$\bar{E} = (1/2)(E_x/G) - \nu(E_x/E_z), \quad \tilde{E} = E_x/E_z, \quad \hat{\alpha} = \alpha_x/\alpha_z.$$

For the specific configuration described in the previous section, the upper and lower surfaces of the beam are traction-free and there is no moment or shear at the free end of the beam. Hence we have the boundary conditions

$$f(\xi, \pm 1) = f_{,\zeta}(\xi, \pm 1) = 0. \tag{31}$$

If we seek asymptotic approximations of the form

$$f = f^{(0)} + \varepsilon^2 f^{(1)} + \cdots + \varepsilon^{2N+2} f^{(N+1)}, \tag{32}$$

substitute (32) into (30), and use the previously obtained expansion for τ in powers of ε^2, we obtain a recurrence relation for $f^{(n)}$ analogous to (11). The details of the required integrations can be found in Duva and Simmonds. These manipulations, along with the imposition of the boundary conditions in (31) and the use of (12a), give

$$\begin{gathered} f^{(0)} = 0 \quad f^{(1)} = 0 \\ f^{(2)}_{,\zeta\zeta\zeta\zeta} = -(\hat{\alpha}\tau^{(1)}_{,\zeta\zeta} + \tau^{(0)}_{,\xi\xi}) = (\tilde{k}\hat{\alpha} - 1)\theta''(\xi)\zeta. \end{gathered} \tag{33}$$

Because $f^{(2)}$ is the first nonzero term in the expansion for f, the dominant axial stress due to the temperature field is

$$\sigma_{xx} = F_{,zz} = O[\varepsilon^2(\tilde{k}\hat{\alpha} - 1)E_x\alpha_z T_0].$$

If the product of $\tilde{k}$ and $\hat{\alpha}$ is unity then the dominant axial stress is of yet higher order. This is consistent with (28b), which shows that, if the material is isotropic, the stresses are identically zero.

In this manner a dimensionless stress function of the form of (32) with a formal error of $O(\varepsilon^{2N+4})$ can be computed for a dimensionless temperature field of the form of (10) with a formal error of $O(\varepsilon^{2N+2})$. Displacements giving rise to a kinematically admissible strain field that is nearby the strain field obtained from these statically admissible stresses can be produced following the program of Duva and Simmonds, and is omitted here. The size of the relative error will be $O((H/l)^{2N+2})$. If the displacements so calculated fail to satisfy the boundary conditions at the ends of the beam, then a full two-dimensional analysis is required there, see Gregory and Wan (1984).

Because of the lack of severe displacement constraints in the configuration chosen, the only stresses arising in the thermally loaded homogeneous beam are the so-called free-body incompatibility stresses (Boley and Weiner) due to the spatial variation of the temperature field. In a laminate, because of the continuity of tractions and displacements at the interface, there is a much stronger coupling between the thermal strains and the stress. This is illustrated in the next section.

APPLICATION TO A LAMINATED BEAM

We consider a simple yet realistic thermally loaded laminated beam consisting of a central, isotropic layer occupying the region $|z| \leqslant H/2$, and two outer, orthotropic layers occupying the regions $H/2 < |z| \leqslant H$; see Fig. 2. The orthotropic material axes are aligned

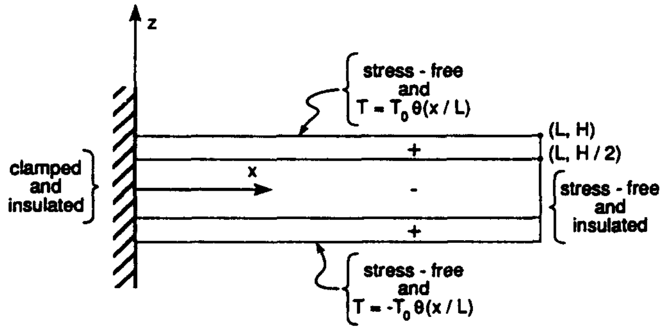

Fig. 2. An undeformed symmetrically laminated rectangular orthotropic beam of length L and height $2H$ subjected to the boundary conditions as shown. The thermal and mechanical bonding at $z = \pm\frac{1}{2}$ is taken to be perfect. It is assumed that a state of plane stress holds and that there is no heat flow perpendicular to the xz-plane.

with the reference axes. Quantities associated with the outer/inner layers are marked with a $+/-$ subscript. We take the conduction coefficients k_{+z}, k_{-z}, and k_{-x} to be equal, as would be the case if both materials were the same continuous fiber-reinforced material with the fibers in the x-direction in the outer layers and perpendicular to the xz-plane in the inner layer. The thermal and mechanical boundary conditions are as described in the previous section. Symmetry allows us to consider only the portion of the beam in the upper half plane, with the conditions

$$T(x,0) = \sigma_{zz}(x,0) = u(x,0) = 0, \tag{34}$$

where u is the displacement in the x-direction.

Our aim is to construct a suitable temperature distribution in each layer following the program decribed in the previous section. Integrating (6) in each layer, we obtain

$$\tau_{\pm} = A_{\pm}(\xi)\zeta + B_{\pm}(\xi) - \varepsilon^2\bar{k}_{\pm}\int_{\zeta_{\pm}}^{\zeta}(\zeta-s)\tau_{\pm,\xi\xi}(\xi,s)\,\mathrm{d}s, \tag{35}$$

where $\bar{k}_{-} = 1$, $\zeta_{+} = 1$ and $\zeta_{-} = 0$. Boundary conditions (7b) give $B_{-} = 0$ and $B_{+} = \theta(\xi) - A_{+}(\xi)$. Thus

$$\tau_{+} = \theta(\xi) + A_{+}(\xi)(\zeta-1) - \varepsilon^2\bar{k}_{+}\left[\int_{1}^{\zeta}(\zeta-s)\tau_{+}(\xi,s)\,\mathrm{d}s\right]_{,\xi\xi} \tag{36a}$$

$$\tau_{-} = A_{-}(\xi)\zeta - \varepsilon^2\left[\int_{0}^{\zeta}(\zeta-s)\tau_{-}(\xi,s)\,\mathrm{d}s\right]_{,\xi\xi}. \tag{36b}$$

The functions of integration $A_{\pm}(\xi)$ are determined by the interface conditions.† With a perfect thermal bond, the continuity of temperature and heat flux at $\zeta = \frac{1}{2}$ are expressed as

$$\tau_{+}(\xi,\tfrac{1}{2}) = \tau_{-}(\xi,\tfrac{1}{2}) \tag{37a}$$

$$\tau_{+,\zeta}(\xi,\tfrac{1}{2}) = \tau_{-,\zeta}(\xi,\tfrac{1}{2}). \tag{37b}$$

Insertion of (36a, b) gives

$$\tau_{\pm} = \tau_{\pm}^{(0)} + \varepsilon^2 I_{\pm}\tau_{,\xi\xi} \tag{38}$$

where

$$\tau_{\pm}^{(0)} = \theta(\xi)\zeta,$$

$$I_{+}\tau = (1-\zeta)\left[\int_{0}^{1/2}s\tau_{-}(\xi,s)\,\mathrm{d}s + \bar{k}_{+}\int_{1/2}^{1}s\tau_{+}(\xi,s)\,\mathrm{d}s\right] + \bar{k}_{+}\int_{\zeta}^{1}(\zeta-s)\tau_{+}(\xi,s)\,\mathrm{d}s,$$

$$I_{-}\tau = \zeta\left[\int_{0}^{1/2}(1-s)\tau_{-}(\xi,s)\,\mathrm{d}s + \bar{k}_{+}\int_{1/2}^{1}(1-s)\tau_{+}(\xi,s)\,\mathrm{d}s\right] - \int_{0}^{\zeta}(\zeta-s)\tau_{-}(\xi,s)\,\mathrm{d}s.$$

If we seek a solution of the form

$$\tau_{\pm} = \tau_{\pm}^{(0)} + \varepsilon^2\tau_{\pm}^{(1)} + \cdots + \varepsilon^{2N}\tau_{\pm}^{(N)}, \tag{39}$$

then (38) yields the recurrence relation

† For a many-layered laminate it is sometimes convenient to write the boundary and interface conditions in matrix form. To do so we would have to write the unknown functions $A_{\pm}(\xi)$ as truncated series in ε^2 analogous to (10).

$$\tau_{\pm}^{(n)} = I_{\pm}\tau_{,\zeta\zeta}^{(n-1)}, \quad n = 1,2,\ldots,N. \tag{40}$$

The application of this relation gives, for $n = 1$,

$$\tau_{+}^{(1)} = (1/24)(1-\zeta)\theta''(\xi)[1-\tilde{k}_{+}(1-4\zeta-4\zeta^2)] \tag{41a}$$

$$\tau_{-}^{(1)} = (1/12)\zeta\theta''(\xi)(1+\tilde{k}_{+}-2\zeta^2). \tag{41b}$$

Note that for $\tilde{k}_{+} = 1$, (41a, b) both reduce to (12a). For arbitrary n, a relation of the form of (12b) holds in each layer. If N-term expressions for τ of the form (10) are obtained in both layers, the relative error will be given by (25), or, in the case that $\tilde{k}_{+} = O(\varepsilon^{-1})$, by (27).

To calculate stresses we will take, consistent with the material model so far, the moduli $E_{+z} = E_{-z} = E_{-x}$ and the coefficients of thermal expansion $\alpha_{+z} = \alpha_{-z} = \alpha_{-x}$. For convenience we will also let the Poisson's ratios $\nu_{+} = \nu_{-} = \nu$. With these choices the symbolism introduced previously can be used; in particular, $\tilde{E} = E_{+x}/E_{+z}$ and $\hat{\alpha} = \alpha_{+x}/\alpha_{+z}$. In terms of the dimensionless stress function $f = F/(L^2E_{+x}\alpha_{+z}T_0)$, the zero traction boundary conditions at $\zeta = 1$ are

$$f_{+}(\xi,1) = f_{+,\zeta}(\xi,1) = 0 \tag{42a}$$

and the zero normal traction–zero axial displacement boundary conditions at $\zeta = 0$ are

$$f_{-}(\xi,0) = \tilde{E}f_{-,\zeta\zeta}(\xi,0)+\varepsilon^2[\tau_{-}(\xi,0)-\nu\tilde{E}f_{-,\xi\xi}(\xi,0)] = 0. \tag{42b}$$

The continuity of tractions and displacements at the interface can be written as

$$f_{+}-f_{-} = 0 \tag{43a}$$

$$f_{+,\zeta}-f_{-,\zeta} = 0 \tag{43b}$$

$$(f_{+}-\tilde{E}f_{-})_{,\zeta\zeta}+\varepsilon^2[\hat{\alpha}\tau_{+}-\tau_{-}-\nu\tilde{E}(f_{+}-f_{-})_{,\xi\xi}] = 0 \tag{43c}$$

$$(f_{+}-\tilde{E}f_{-})_{,\zeta\zeta\zeta}+\varepsilon^2\{\hat{\alpha}\tau_{+}-\tau_{-}+[(\hat{E}_{+}-\nu\tilde{E})f_{+}-(\hat{E}_{-}-\nu\tilde{E})f_{-}]_{,\xi\xi}\}_{,\zeta} = 0, \tag{43d}$$

where $\hat{E}_{\pm} = E_{\pm x}/G_{\pm}$. The last two conditions are derived by first differentiating the displacement continuity conditions with respect to x to obtain strain continuity conditions. The stress–strain relations given in the Appendix are then used to express, in each layer, the strains in terms of stresses and the temperature, with the stresses expressed in terms of the dimensionless stress function f. Also, (43a) is used to eliminate a pair of terms that would otherwise appear in (43c).

If we seek a solution of the form of (32) in each layer, then (30) and conditions (42a, b) and (43a, b, c, d) provide the recurrence relations necessary to generate the unknown functions $f_{\pm}^{(n)}$. In particular, we find $f_{\pm}^{(0)} = 0$ and

$$f_{+}^{(1)} = \frac{\theta(\xi)(1-\hat{\alpha})}{6(1+7\tilde{E})}(\zeta-1)^2(\zeta+2) \tag{44a}$$

$$f_{-}^{(1)} = \frac{\theta(\xi)(1-\hat{\alpha})}{6(1+7\tilde{E})}\zeta(3-7\zeta^2). \tag{44b}$$

To lowest order, the stresses at the interface computed from (44a, b) are

$$\sigma_{zz} = (5/8)\varepsilon^2\theta''\Sigma, \tag{45a}$$

$$\sigma_{xz} = -(9/4)\varepsilon\theta'\Sigma, \tag{45b}$$

$$\sigma_{+xx} = 3\theta\Sigma, \tag{45c}$$

$$\sigma_{-xx} = -21\theta\Sigma, \tag{45d}$$

where

$$\Sigma = \frac{T_0 E_{+x}\alpha_{+z}(1-\hat{\alpha})}{6(1+7\tilde{E})}.$$

The size of the relative error for these approximations is $O(H^2/l^2)$.

CONCLUSIONS

We have laid out a program to generate approximate two-dimensional temperature fields and concomitant stress fields in a beam subjected to thermal loads. These fields can be made as accurate as desired. This program is applicable even if the material properties, such as the thermal conductivity, are severely anisotropic, or if the beam is layered. Our error estimates are based on the Prager–Synge hypercircle method that requires prescribed thermal and mechanical conditions at the ends of the beam to be satisfied by the approximate solutions we construct. If this is not so, there will be end effects and a full two-dimensional treatment is required to insure accuracy everywhere.

We emphasize again that, contrary to the assertions made or impressions left by some authors, the elementary theory of thermoelastic beams, interpreted properly, is *not* based on assuming variations of temperature, displacements or stresses through the thickness. Our approach shows one may infer these thickness distributions—even if they have kinks at interfaces, as is the usual case with laminates—after the one-dimensional beam equations have been solved.

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APPENDIX

The relations used to derive (28a) and (43c, d) are

$$\varepsilon_{xx} = u_{,x} = \frac{1}{E_x}(\sigma_{xx} - \nu\tilde{E}\sigma_{zz}) + \alpha_x T = \frac{1}{E_x}(F_{,zz} - \nu\tilde{E}F_{,xx}) + \alpha_x T, \tag{A1}$$

$$\varepsilon_{zz} = w_{,z} = \frac{1}{E_z}(\sigma_{zz} - \nu\sigma_{xx}) + \alpha_z T = \frac{1}{E_z}(F_{,xx} - \nu F_{,zz}) + \alpha_z T, \tag{A2}$$

$$2\varepsilon_{xz} = w_{,x} + u_{,z} = \frac{1}{G}\sigma_{xz} = -\frac{1}{G}F_{,xz}, \tag{A3}$$

where u and w are displacements in the x- and z-directions, respectively, and, in an alternative notation for orthotropic materials,

$$\nu = \nu_{xz} = \nu_{zx}E_z/E_x.$$

Note that the value of $\tilde{E} = E_x/E_z$ depends on the layer in question.